On conserved operator quantities in quantum field theory

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Abstract

Conserved operator quantities in quantum field theory can be defined via the Noether theorem in the Lagrangian formalism and as generators of some transformations. These definitions lead to generally different conserved operators which are suitable for different purposes. Some relations involving conserved operators are analyzed.

1. Introduction

There are two approaches for introduction of conserved operator quantities in quantum field theory. The first one is based on the Lagrangian formalism and defines them via the first Noether theorem as conserved operators corresponding to smooth transformations living invariant the action integral of an investigated system; these are the canonical conserved operators. The second set of conserved operators consists of generators of some transformations of state vectors (and observables). Since these operators are of pure mathematical origin, we call them mathematical conserved quantities (operators). The present paper is devoted to a discussion of some relations between the mentioned two kinds of conserved quantities in quantum field theory. It is pointed that the two types of conserved operators are generally different and may coincide on some subspace of the system's Hilbert space of states.

The present work generalizes part of the results of [**?**, **?**] and may be considered as a continuation of these papers.

In what follows, we suppose that there is given a system of quantum fields, described via field operators $\varphi_i(x)$, $i = 1, \dots, n \in \mathbb{N}$, $x \in M$ over the 4-dimensional Minkowski spacetime M endowed with standard Lorentzian metric tensor $\eta_{\mu\nu}$ with signature $(+\,-\,-\,-)$.[1] The system's Hilbert space of states is denoted by $\mathcal{F}$ and all considerations are in Heisenberg picture of motion if the opposite is not stated explicitly. The Greek indices $\mu, \nu, \dots$ run from 0 to $3 = \dim M - 1$ and the Einstein's summation convention is assumed over indices repeated on different levels. The coordinates of a point $x \in M$ are denoted by x^μ, $\boldsymbol{x} := (x^1, x^2, x^3)$, $\mathrm{d}^3\boldsymbol{x} := \mathrm{d}x^1\,\mathrm{d}x^2\,\mathrm{d}x^3$, and the derivative with respect to x^μ is $\frac{\partial}{\partial x^\mu} =: \partial_\mu$. The imaginary unit is denoted by i and $\hbar$ and c stand for the Planck's constant (divided by 2π) and the velocity of light in vacuum, respectively.

2. Canonical conserved quantities

Suppose a system of *classical* fields $\varphi_i(x)$, $i = 1, \dots, n \in \mathbb{N}$, over the Minkowski spacetime M, $x \in M$, is described via a Lagrangian L depending on them and their first partial derivatives $\partial_\mu \varphi_i(x) = \frac{\partial \varphi_i(x)}{\partial x^\mu}$, $\{x^\mu\}$ being the (local) coordinates of $x \in M$, i.e. $L = L(\varphi_j(x), \partial_\nu \varphi_i(x))$. Here and henceforth the Greek indices $\mu, \nu, \dots$ run from 0 to $\dim M - 1 = 3$ and the Latin indices $i, j, \dots$ run from 1 to some integer n. The equations of motion for $\varphi_i(x)$, known as the *Euler-Lagrange equations*, are[2] $\frac{\partial L}{\partial \varphi_i(x)} - \frac{\partial}{\partial x^\mu}\Big(\frac{\partial L}{\partial(\partial_\mu \varphi_i(x))}\Big) = 0$ and are derived from the variational principle of stationary action, known as the *action principle* (see, e.g, [**?**, § 1], [**?**, § 67], [**?**, pp. 19–20]).

The (first) Noether theorem [**?**, § 2] says that, if the action's variation is invariant under C^1 transformations

$$\pi^{i\mu} := \frac{\partial L}{\partial(\partial_\mu \varphi_i(x))},$$

are conserved in a sense that their divergences vanish, viz.

$$\partial_\mu \theta^\mu_{(\alpha)}(x) = 0.$$

$\varphi_i(x) \mapsto \varphi_i^\omega(x^\omega) \quad \varphi_i^\omega(x^\omega)|_{\omega=\mathbf{0}} = \varphi_i(x)$

[1] The quantum fields should be regarded as operator-valued distributions (acting on a relevant space of test functions) in the rigorous mathematical setting of Lagrangian quantum field theory. This approach will be considered elsewhere.

[2] In this paper the Einstein's summation convention over indices appearing twice on different levels is assumed over the whole range of their values.

depending on $s \in \mathbb{N}$ independent real parameters $\omega^{(1)}, \dots, \omega^{(s)}$, then the quantities ('Noether currents') $\theta^{\mu}_{(\alpha)}(x) := -\pi^{i\mu}\Big\{ \frac{\partial \varphi^{\omega}_{i}(x^{\omega})}{\partial \omega^{(\alpha)}}\Big|_{\omega=0} - (\partial_\nu \varphi_i(x)) \frac{\partial x^{\omega\,\nu}}{\partial \omega^{(\alpha)}}\Big|_{\omega=0} \Big\} - L(x) \frac{\partial x^{\omega\,\mu}}{\partial \omega^{(\alpha)}}\Big|_{\omega=0}, where \alpha = 1, \dots, s$ and $\pi^{i\mu} := \frac{\partial L}{\partial(\partial_\mu \varphi_i(x))}, are conserved in a sense that their divergences vanish, viz. \partial_\mu \theta^{\mu}_{(\alpha)}(x) = 0.$

Respectively, the quantities

$$C_{(\alpha)}(x) := \frac{1}{c} \int \theta^{0}_{(\alpha)} \, \mathrm{d}^3 \boldsymbol{x}, \tag{2.2}$$

which in fact may depend only on x^0, are conserved in a sense that

$$\frac{\partial C_{(\alpha)}(x)}{\partial x^0} = 0 \tag{2.3}$$

and hence $\partial_\mu C_{(\alpha)} = 0$. The functions (constants) $C_{(\alpha)}$ are called *canonical (Noetherian, dynamical) conserved quantities* corresponding to the symmetry transformations (2) of the system considered.

Let us turn now our attention to a system of *quantum* fields represented by *field operators* $\varphi_i(x) \colon \mathcal{F} \to \mathcal{F}$, $i = 1, \dots, n \in \mathbb{N}$, acting on the system's Hilbert space $\mathcal{F}$ of states and described via a Lagrangian $\mathcal{L} = \mathcal{L}(x) = \mathcal{L}(\varphi_i(x), \partial_\mu \varphi_j(x))$. Supposed the system's action integral is invariant under the C^1 transformations (2). As a consequence of that supposition, one may expect the *operators* (2), with $\pi^{i\mu}$ defined via (2), to be conserved, i.e. the equations (2) to be valid. However, at this point two problems arise: (i) what is the meaning of the derivatives in (2) as $\partial_\mu \varphi_i(x)$ is *operator*, not a classical function? and (ii) in what order one should write the operators compositions in (2), e.g. shall we write $\pi^{i\mu} \circ \partial_\nu \varphi_i(x)$ or $\partial_\nu \varphi_i(x) \circ \pi^{i\mu}$? Usually [?, ?, ?] these problems are solved by (implicitly) adding to the theory additional assumptions concerning the operator ordering in (2) and meaning of derivatives with respect to operator-valued arguments.[3] In the work [?] we demonstrated that there is only one problem connected with a suitable definition of derivatives relative to operator-valued arguments and all other results follow directly from the (Schwinger's) action principle. The main point is that such derivatives are mappings form (a subset of) the space $\{\mathcal{F} \to \mathcal{F}\}$ of operators on $\mathcal{F}$ into $\{\mathcal{F} \to \mathcal{F}\}$ rather than operators $\mathcal{F} \to \mathcal{F}$. In particular, we have

$$\pi^{i\mu}(x) := \frac{\partial L}{\partial(\partial_\mu \varphi_i(x))} \colon \{\mathcal{F} \to \mathcal{F}\} \to \{\mathcal{F} \to \mathcal{F}\}. \tag{2.4}$$

For details and the rigorous definition of a derivative (of polynomial or convergent power series) relative to operator-valued argument, the reader is referred to [?]. Accepting (2.4), we can write the quantum field analogue of (2), i.e. the 'Noether's current operators", as

$$\theta^{\mu}_{(\alpha)}(x) := -\sum_i \pi^{i\mu}(x) \Big(\frac{\partial \varphi^{\omega}_{i}(x^{\omega})}{\partial \omega^{(\alpha)}}\Big|_{\omega=0} \Big) + \sum_{i,\nu} \pi^{i\mu}(x) \big(\partial_\nu \varphi_i(x)\big) \frac{\partial x^{\omega\,\nu}}{\partial \omega^{(\alpha)}}\Big|_{\omega=0} - L(x) \frac{\partial x^{\omega\,\mu}}{\partial \omega^{(\alpha)}}\Big|_{\omega=0}, \tag{2.5}$$

which immediately leads to the conservation laws (2) and (2.3). The quantities (2.2), with $\theta^{\mu}_{(\alpha)}$ given by (2.5), are called the *canonical (Noetherian, dynamical) conserved operators* corresponding to the symmetry transformations (2).

[3] E.g., derivatives like the ones in (2) are calculated according to the rules of classical analysis of commuting variables by preserving the relative order of all terms in the Lagrangian. As pointed in [?], this rule corresponds to field variations proportional to the identity mapping $\mathsf{id}_{\mathcal{F}}$ of $\mathcal{F}$.

We end this section by the remark that the momentum, (total) angular momentum, and charge conserved operators are generated respectively by the transformation:

$$x \mapsto x + b \qquad \varphi_i(x) \mapsto \varphi_i(x) \tag{2.6a}$$

$$x^{\varkappa} \mapsto x^{\varkappa} + \varepsilon^{\varkappa\nu} x_\nu \quad \varphi_i(x) \mapsto \varphi_i(x) + \frac{1}{2} I^j_{i\mu\nu} \varepsilon^{\mu\nu} \varphi_j(x) + \cdots \tag{2.6b}$$

$$x \mapsto x \qquad \varphi_i(x) \mapsto \mathrm{e}^{\frac{q}{i\hbar c}\lambda} \varphi_i(x), \tag{2.6c}$$

where $b \in M$, $\varepsilon^{\mu\nu} = -\varepsilon^{\nu\mu} \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ are the parameters of the corresponding transformations, x_μ are the covariant coordinates of $x \in M$, the numbers $I^j_{i\mu\nu} = -I^j_{i\nu\mu}$ characterize the behaviour of the field operators under rotations, and the dots stand for higher order terms in $\varepsilon^{\mu\nu}$.

3. On observer dependence of state vectors and observables

Let two observers O and O' investigate one and the same system of quantum fields. The quantities relative to O' will be denoted as those relative to O by adding a prime to their kernel symbols. The transition $O \mapsto O'$ implies the changes

$$x \mapsto x' = L(x) \tag{3.1}$$

(of the coordinates) of a spacetime point $x = (x^0, x^1, x^2, x^3) \in M$ and

$$\mathcal{X}(x) \mapsto \mathcal{X}'(x') = \Lambda(\mathcal{X}(x)) \tag{3.2}$$

of a state vector $\mathcal{X}(x) \in \mathcal{F}$ of system of quantum fields $\varphi_i(x)$.[4] Requiring preservation of the scalar products in $\mathcal{F}$ under the change $O \mapsto O'$, which physically corresponds to preservation of probability amplitudes, we see that Λ is a *unitary* operator,

$$\Lambda^{-1} = \Lambda^\dagger \tag{3.3}$$

where the dagger $\dagger$ denotes Hermitian conjugation (i.e., in mathematical terms, $\Lambda^\dagger$ is the adjoint to Λ operator).

Let $\mathbf{A}$ be a dynamical variable and $\mathcal{A}(x) : \mathcal{F} \to \mathcal{F}$ be the corresponding to it observable. The change $O \mapsto O'$ entails $\mathcal{A}(x) \mapsto \mathcal{A}(L(x))$. Supposing preservation of the mean (expectation) values (and the matrix elements of $\mathbf{A}$ (or $\mathcal{A}(x)$)) in states with finite norm under the change $O \mapsto O'$, we get

$$\mathcal{A}(L(x)) = (\Lambda^\dagger)^{-1} \circ \mathcal{A}(x) \circ \Lambda^{-1} = \Lambda \circ \mathcal{A}(x) \circ \Lambda^{-1}. \tag{3.4}$$

As explained in [**?**, sect. 4] or in [**?**, **?**], the field operators $\varphi_i(x)$ undergo more complicated change when one passes from O to O':

$$\varphi_i(x) \mapsto \sum_j (S^{-1})_i^{\;j}(L) \varphi_j(x) = \Lambda \circ \varphi_i(x) \circ \Lambda^{-1} \tag{3.5}$$

where the depending on L matrix $S = S(L) = [(S^{-1})_i^{\;j}(L)]$ characterizes the transformation properties of any particular field (e.g. scale or vector one) under $O \mapsto O'$ and is such that $S(L)|_{L=\mathsf{id}_M}$ is the identity matrix of relevant size.

[4] It is inessential for the following whether L (Λ) is an element (of a representation) of the Poincaré group or not; the former case is realized when O and O' are inertial observers.

4. Transformations with Hermitian generators

Let $\omega^1, \dots, \omega^s$, $s \in \mathbb{N}$, be real independent parameters and $\omega := (\omega^1, \dots, \omega^s) \in \mathbb{R}^s$. Suppose the changes (3.1) and (3.2) depend on ω and

$$
\begin{aligned}
& x \mapsto x' = L^\omega(x) = x^\omega(x) \quad x^\omega(x)|_{\omega=0} = x \\
& \Lambda = \Lambda^\omega = \exp\Big\{ \frac{\eta}{\mathrm{i}\hbar} \sum_{\alpha=1}^{s} \omega^\alpha \mathcal{J}_\alpha^{\mathrm{m}} \Big\},
\end{aligned}
\tag{4.1}
$$

where the operators $\mathcal{J}_\alpha^{\mathrm{m}} \colon \mathcal{F} \to \mathcal{F}$ are Hermitian,

$$
(\mathcal{J}_\alpha^{\mathrm{m}})^\dagger = \mathcal{J}_\alpha^{\mathrm{m}}, \tag{4.2}
$$

which ensures the validity of (3.3), and the particular choice of the constant $\eta \in \mathbb{R} \setminus \{0\}$ depends on what physical interpretation of $\mathcal{J}_\alpha^{\mathrm{m}}$ one intends to get.

Differentiating (3.4) and (3.5) with respect to ω^α and setting $\omega = 0$, we rewrite them in differential form respectively as

$$
\eta[\mathcal{A}(x), \mathcal{J}_\alpha^{\mathrm{m}}]_{\_} = -\mathrm{i}\hbar \frac{\partial \mathcal{A}(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial \omega^\alpha}\Big|_{\omega=0} \tag{4.3}
$$

$$
\eta[\varphi_i(x), \mathcal{J}_\alpha^{\mathrm{m}}]_{\_} = \mathrm{i}\hbar \sum_j I_{i\alpha}^j \varphi_j(x) - \mathrm{i}\hbar \frac{\partial \varphi_i(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial \omega^\alpha}\Big|_{\omega=0} \tag{4.4}
$$

where $I_{i\alpha}^j := \frac{\partial S_i^{\;j}(L^\omega)}{\partial \omega^\alpha}\Big|_{\omega=0}$, i.e. $S_i^{\;j}(L^\omega) = \delta_i^j + \sum_\alpha I_{i\alpha}^j \omega^\alpha + \cdots$ with δ_i^j being the Kroneker deltas and the dots denoting higher order terms in ω, and $[\mathcal{A}, \mathcal{B}]_{\_} := \mathcal{A} \circ \mathcal{B} - \mathcal{B} \circ \mathcal{A}$ is the commutator of operators $\mathcal{A}, \mathcal{B} \colon \mathcal{F} \to \mathcal{F}$.

In particular, to describe the quantum analogue of the transformations (2.6), in (4.1) we have to make respectively the replacements:

$$
\begin{aligned}
&\omega^\alpha \mapsto b^\mu && x^\omega \mapsto x^b = x + b && \eta \mapsto -1 && \mathcal{J}_\alpha^{\mathrm{m}} \mapsto \mathcal{P}_\mu^{\mathrm{t}} && \text{(4.5a)}\\
&\omega^\alpha \mapsto \varepsilon^{\mu\nu} \quad (\mu < \nu) && x^{\omega\,\mu} \mapsto x^{\varepsilon\,\mu} = x^\mu + \varepsilon^{\mu\nu} x_\nu && \eta \mapsto +1 && \mathcal{J}_\alpha^{\mathrm{m}} \mapsto \mathcal{M}_{\mu\nu}^{\mathrm{r}} && \text{(4.5b)}\\
&\omega^\alpha \mapsto \lambda && x^\omega \mapsto x^\lambda = x && \eta \mapsto \frac{q}{c} && \mathcal{J}_\alpha^{\mathrm{m}} \mapsto \mathcal{Q}^{\mathrm{p}}, && \text{(4.5c)}
\end{aligned}
$$

so that $\frac{\partial x^{\omega\,\varkappa}}{\partial \omega^\alpha}\big|_{\omega=0}$ reduces to $\delta_\mu^\varkappa$, $(\delta_\mu^\varkappa x_\nu - \delta_\nu^\varkappa x_\mu)$, and $0 \in \mathbb{R}$, respectively. The operators $\mathcal{P}_\mu^{\mathrm{t}}$, $\mathcal{M}_{\mu\nu}^{\mathrm{r}}$, and $\mathcal{Q}^{\mathrm{p}}$ are the translation (mathematical) momentum operator, total rotational (mathematical) angular momentum operator, and constant phase transformation (mathematical) charge operator, respectively. In these cases, the equations (4.4) are known as the Heisenberg equations/relations for the operators mentioned [**?**, **?**, **?**] . For that reason, it is convenient to call (4.4) *Heisenberg equations/relations* (for the operators $\mathcal{J}_\alpha^{\mathrm{m}}$) in the general case.

The transformations (3.1) and (3.5), defined by the choice (4.1), are the *quantum* observer-transformation version of (2). For that reason, one can expect the (spacetime constant) operators $\mathcal{J}_\alpha^{\mathrm{m}}$ to play, in some sense, a role similar to the conserved operators (2.5); we shall call $\mathcal{J}_\alpha^{\mathrm{m}}$ *mathematical conserved operators* corresponding to the transformations (3.1) and (3.5) under the choices (4.1).

Suppose there exist operators $\mathcal{J}_\alpha^{\mathrm{QM}}$, where QM stands for quantum mechanics[5], gener-

[5] This notation reminds only some analogy with quantum mechanics. If one identifies $\mathcal{F}$ with the Hilbert space of this theory and makes some other assumptions, (part of) the generators $\mathcal{J}_\alpha^{\mathrm{QM}}$ will coincide with similar objects in quantum mechanics. However, as the Hilbert spaces of quantum field theory and quantum mechanics are different, the corresponding operators in these theories cannot be identified. See similar remarks in [**?**, **?**] concerning the momentum and angular momentum operators, respectively.

ating the change $\mathcal{X}(x) \mapsto \mathcal{X}(x')$, i.e. such that ($|\eta|$ is the absolute value of η)

$$\mathcal{X}(x) \mapsto \mathcal{X}(x') = \Lambda^{\mathrm{QM}}(\mathcal{X}(x)) := \exp\Big\{ \frac{|\eta|}{\mathrm{i}\hbar} \sum_{\alpha=1}^{s} \omega^\alpha \mathcal{J}_\alpha^{\mathrm{QM}} \Big\} (\mathcal{X}(x)). \tag{4.6}$$

Note that $\mathcal{J}_\alpha^{\mathrm{QM}}$ (as well as $\mathcal{J}_\alpha^{\mathrm{m}}$) may depend on x; for instance, the changes $x \mapsto x^\omega$ defined via (4.5a)–(4.5c) entail (4.6) with respectively ($\mathsf{id}_{\mathcal{F}}$ is the identity mapping of $\mathcal{F}$)

$$\mathcal{J}_\alpha^{\mathrm{QM}} \mapsto \mathcal{P}_\mu^{\mathrm{QM}} = \mathrm{i}\hbar\partial_\mu \tag{4.7a}$$

$$\mathcal{J}_\alpha^{\mathrm{QM}} \mapsto \mathcal{M}_{\mu\nu}^{\mathrm{QM}} = \mathrm{i}\hbar(x_\mu\partial_\nu - x_\nu\partial_\mu) \tag{4.7b}$$

$$\mathcal{J}_\alpha^{\mathrm{QM}} \mapsto \mathcal{Q}_\mu^{\mathrm{QM}} = \mathrm{e}^{\frac{q}{\mathrm{i}\hbar c}\lambda} \,\mathsf{id}_{\mathcal{F}}. \tag{4.7c}$$

The transformation (4.6) implies the changes

$$\mathcal{A}(x) \mapsto \mathcal{A}(x') = \Lambda^{\mathrm{QM}} \circ \mathcal{A}(x) \circ (\Lambda^{\mathrm{QM}})^{-1}. \tag{4.8}$$

$$\varphi_i(x) \mapsto \sum_j (S^{-1})_i^{\,j}(L)\varphi_j(x') = \Lambda^{\mathrm{QM}} \circ \varphi_i(x) \circ (\Lambda^{\mathrm{QM}})^{-1} \tag{4.9}$$

which, in differential form, entail

$$|\eta| [\mathcal{A}(x), \mathcal{J}_\alpha^{\mathrm{QM}}]_- = -\mathrm{i}\hbar \frac{\partial \mathcal{A}(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial\omega^\alpha}\Big|_{\omega=0} \tag{4.10}$$

$$|\eta| [\varphi_i(x), \mathcal{J}_\alpha^{\mathrm{QM}}]_- = \mathrm{i}\hbar \sum_j I_{i\alpha}^{j} \varphi_j(x) - \mathrm{i}\hbar \frac{\partial \varphi_i(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial\omega^\alpha}\Big|_{\omega=0} \tag{4.11}$$

Comparing these equations with (4.3) and (4.4), we find

$$[\mathcal{A}(x), \mathcal{J}_\alpha^{\mathrm{m}} - \operatorname{sign}\eta\, \mathcal{J}_\alpha^{\mathrm{QM}}]_- = 0 \tag{4.12a}$$

$$[\varphi_i(x), \mathcal{J}_\alpha^{\mathrm{m}} - \operatorname{sign}\eta\, \mathcal{J}_\alpha^{\mathrm{QM}}]_- = 0 \tag{4.12b}$$

where $\operatorname{sign}\eta := \eta/|\eta| \in \{-1. +1\}$ is the sign of $\eta \in \mathbb{R}\setminus\{0\}$. If we admit (4.12a) to hold for *every* $\mathcal{A}(x)\colon \mathcal{F} \to \mathcal{F}$, the Schur's lemma[6] implies

$$\mathcal{J}_\alpha^{\mathrm{m}} = \operatorname{sign}\eta\, \mathcal{J}_\alpha^{\mathrm{QM}} + j_\alpha \,\mathsf{id}_{\mathcal{F}}, \tag{4.13}$$

where j_α are real numbers (with the same dimension as the eigenvalues of $\mathcal{J}_\alpha^{\mathrm{m}}$).

5. Discussion

Following the opinion established in the literature[7], the identification

$$C_{(\alpha)} = \mathcal{J}_\alpha^{\mathrm{m}} \tag{5.1}$$

may seem 'natural' *prima facie* but, generally, it is unacceptable as its l.h.s. comes out from the Lagrangian formalism (via (2.5) and (2.2)), while its r.h.s. originates from pure mathematical (geometrical) considerations and is suitable for the axiomatic quantum field theory [**?**, **?**].

As an equality weaker than (5.1), the Heisenberg relations (4.4) with $C_{(\alpha)}$ for $\mathcal{J}_\alpha^{\mathrm{m}}$ can be assumed:

$$\eta[\varphi_i(x), C_{(\alpha)}]_- = \mathrm{i}\hbar \sum_j I_{i\alpha}^{j} \varphi_j(x) - \mathrm{i}\hbar \frac{\partial \varphi_i(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial\omega^\alpha}\Big|_{\omega=0}. \tag{5.2}$$

[6] See, e.g, [**?**, appendix II], [**?**, sec. 8.2], [**?**, ch. 5, sec. 3].
[7] See also the papers [**?**, **?**] in which the momentum and angular momentum are analyzed.

However, these equations as well as (5.1) are external to the Lagrangian formalism by means of which the canonical conserved operators are defined. As discussed in [**?**, § 68] on particular examples, the validity of the equations (5.2) should be checked for any particular Lagrangian and they express (in the sense explained in *loc. cit.*) the relativistic covariance of the Lagrangian quantum field theory.

Generally the equation (4.3) with $C_{(\alpha)}$ for $\mathcal{J}_\alpha^{\mathrm{m}}$, viz.

$$\eta[\mathcal{A}(x), C_{(\alpha)}]_{-} = -\mathrm{i}\hbar \frac{\partial \mathcal{A}(x)}{\partial x^\mu} \frac{\partial x^{\omega\,\mu}}{\partial \omega^\alpha}\Big|_{\omega=0}, \tag{5.3}$$

cannot hold; a counterexample being the choice of $\mathcal{A}(x)$ and $C_{(\alpha)}$ as the momentum and angular momentum operators (or *vice versa*). If (5.3) happens to be valid for operators $\mathcal{A}(x)$ forming an irreducible representation of some group, then, by virtue of (5.3) and (4.3), the Schur's lemma implies

$$C_{(\alpha)} = \operatorname{sign}\eta\, \mathcal{J}_\alpha^{\mathrm{m}} + i_\alpha \,\mathsf{id}_{\mathcal{F}} = \operatorname{sign}\eta\, \mathcal{J}_\alpha^{\mathrm{QM}} + (i_\alpha + j_\alpha)\,\mathsf{id}_{\mathcal{F}} \tag{5.4}$$

for some real numbers i_α (see also (4.13)).

Let a vector $\mathcal{X} \in \mathcal{F}$ represents a state of the system of quantum fields considered. It is a spacetime-constant vector as we are working in Heisenberg picture of motion. Consequently, we have $\mathcal{X}(x) = \mathcal{X}(x')$ which, when combined with (4.6), entails

$$\mathcal{J}_\alpha^{\mathrm{QM}}(\mathcal{X}) = 0. \tag{5.5}$$

So, applying (4.13) to $\mathcal{X}$, we get

$$\mathcal{J}_\alpha^{\mathrm{m}}(\mathcal{X}) = j_\alpha\,\mathcal{X}. \tag{5.6}$$

If one intends to interpret $\mathcal{J}_\alpha^{\mathrm{m}}$ as the conserved canonical operators $C_{(\alpha)}$ (see the possible equality (5.1)), then one should interpret j_α as the mean (expectation) value of $C_{(\alpha)}$, which will be the case if

$$C_{(\alpha)}(\mathcal{X}) = j_\alpha\,\mathcal{X}. \tag{5.7}$$

(Notice, (5.7) and (5.4) are compatible iff $i_\alpha = 0$.) The equations (5.6) and (5.7) imply

$$C_{(\alpha)}|_{\mathcal{D}_j} = \mathcal{J}_\alpha^{\mathrm{m}}|_{\mathcal{D}_j}. \tag{5.8}$$

where

$$\mathcal{D}_j := \{\mathcal{X} \in \mathcal{F} : C_{(\alpha)}(\mathcal{X}) = j_\alpha\,\mathcal{X}\}. \tag{5.9}$$

Generally the set $\mathcal{D}_j$ is a proper subset of $\mathcal{F}$ and hence (5.8) is weaker than (5.1); if a basis of $\mathcal{F}$ can be formed from vectors in $\mathcal{D}_j$, then (5.8) and (5.1) will be equivalent. But, in the general case, equations (5.2) and (4.4) lead only to

$$[\varphi_i(x), C_{(\alpha)}]_{-} = [\varphi_i(x), \mathcal{J}_\alpha^{\mathrm{m}}]_{-} \qquad \Big(= \frac{1}{\eta}\mathrm{i}\hbar \sum_j I_{i\alpha}^j \varphi_j(x) - \frac{1}{\eta}\mathrm{i}\hbar \frac{\partial\varphi_i(x)}{\partial x^\mu}\frac{\partial x^{\omega\,\mu}}{\partial\omega^\alpha}\Big|_{\omega=0}\Big), \tag{5.10}$$

but not to (5.1).

Ending this section, we note that the equality $C_{(\alpha)} = \mathcal{J}_\alpha^{\mathrm{QM}}$ is unacceptable as, in view of (5.5), it leads to identically vanishing eigenvalues of $C_{(\alpha)}$.

6. Conclusion

In this work we have analyzed two types of conserved operator quantities in quantum field theory, viz. the ones arising from the (first) Noether theorem in the framework of Lagrangian formalism and conserved operators having pure mathematical origin as generators of some transformations (and having natural place in the axiomatic approach). These operators are generally different and their equality is a problem which is external to the Lagrangian formalism and may be considered as possible subsidiary restrictions to it. However, using the arbitrariness (4.13) in the mathematical conserved operators, both types of conserved operators can be chosen to coincide on the set (5.9). As weaker conditions additionally imposed on the Lagrangian formalism, one can require the equality (5.10) between the commutators of the field operators and conserved operators. As it is known [**?**], the Heisenberg relations (5.2) are equations relative to the field operators, while (4.4) are identities with respect to them.